Teachers usually present addition and subtraction at the same time to first and second graders. Many teachers have observed, however, that students find addition easier and more natural than subtraction. Children struggle with subtraction even when they learn “fact families” that ostensibly help them understand the relationship between addition and subtraction. Given that children continue to find subtraction difficult despite the use of time-honored practices, we suggest that teachers de-emphasize fluency in subtraction until their students become fluent in addition.

Why is subtraction difficult in first and second grade? Piaget said that subtraction is more difficult than addition because it involves a negative action that is a secondary construction of addition (Piaget 1980). He also stated that children initially think and perceive in only positive terms. For example, when young children see a red ball, they first think of it only as “red” and “a ball.” A considerable time later, they are able to think of the same ball as not blue, not green, not a cup, and not a shoe.

With this theoretical insight, one of the authors (Kamii 1985) decided to ask a few subtraction questions in the first-grade classroom in which she was working. It was the middle of the school year, and these children had never been formally taught subtraction. The first question was “What's 10 take away 5?” The entire class immediately shouted, “Five!” Kamii asked, “What's 4 take away 2?” and all the students shouted, “Two!” while jumping up and down. The next question was “What's 7 take away 4?” The children began to count on their fingers.

These specific questions were selected because by October, 96 percent of the class could instantly give answers to $5 + 5$ and $2 + 2$, but no one knew the answer to $4 + 3$. As many teachers know, “doubles” such as $5 + 5$ and $2 + 2$ are much easier to learn than combinations such as $4 + 3$. These first graders verified Piaget’s (1980) statement that subtraction is a later, secondary construction. Once children’s knowledge of a sum is solid, the related subtraction is easy for them.

Recently, we conducted more systematic research to test the hypothesis that children’s knowledge of differences depends on their knowledge of sums (Kamii, Lewis, and Kirkland 2001). We asked children to solve related addition and subtraction problems such as $8 + 2$ and $10 - 8$. If we found that fluency in subtraction was unrelated to fluency in addition, we would conclude that knowledge of differences and knowledge of sums are independent of each other. If, however, we found that fluency in subtraction was dependent on fluency in addition, we would infer that children deduce differences from their knowledge of sums.

Method
We interviewed twenty-one first graders from one class and thirty-eight fourth graders from two classes individually in a constructivist “school within a school” located in an upper-middle-class suburb. We selected the two grade levels because one of us was working in these classrooms and the grade levels seemed appropriate for our purpose. The children in this “school within a school” did not use any textbooks, workbooks, or worksheets. Instead, they learned arithmetic through three kinds of activities: situations in daily living (such as voting), word problems, and mathematics games. As Kamii (1994, 2000, forthcoming) describes, the children were not taught how to do anything. They were not taught conventional algorithms such as “carrying” and “borrowing”; instead, they invented their own computational procedures as they solved word problems. The teacher did not give them timed tests or pressure them to produce answers.
with speed. The students simply played many mathematics games and remembered sums and differences from playing these games every day. We de-emphasized subtraction, especially in the first two grades, as we became aware of the findings described in this article.

The interviews were part of the routine assessment conducted at the beginning and end of each school year with a four-page form. The form contained about seventy one-, two-, and three-digit computational problems involving all four operations. Both the student and the interviewer had a copy of the form that included all the questions in a column on the left-hand side. The interviewer asked the child to give the answer orally to each question and slide a ruler down to the next question. The interviewer recorded what the child said and occasionally asked how the child had obtained an answer. One dot per second indicated the child’s reaction time.

We used only two of the four pages of the interview form with the first graders and told the fourth graders that they could skip any problem that was too difficult. We did not tell the children that speed mattered in this assessment.

We selected four pairs of questions from the first graders’ end-of-the-year interviews: 8 + 2 and 10 – 8; 6 + 6 and 12 – 6; 4 + 6 and 10 – 6; and 4 + 4 and 8 – 4. All the addition problems appeared at the beginning of the interview, and the subtraction problems were toward the end.

For the fourth graders, we added five pairs of more difficult questions to the form: 5 + 3 and 8 – 5; 5 + 6 and 11 – 5; 7 + 4 and 11 – 4; 8 + 5 and 13 – 5; and 7 + 8 and 15 – 8. The fourth-grade results are from the beginning-of-the-year interviews.

**Results**

We categorized all the responses as “Successful” or “Unsuccessful.” To be included in the “Successful” (fluent) category, children had to give the correct answer within three seconds; all other responses were “Unsuccessful.” Most of the children in the “Unsuccessful” category produced correct answers but took more than three seconds to think or count, sometimes with the aid of their fingers. The subtraction problems were sometimes solved by counting up and sometimes by counting down.

**First grade**

**Figure 1** shows the first-grade results. The four tables indicate that addition was much easier than subtraction. Of the twenty-one students, 91% (43% + 48%), 96% (48%+ 48%), 57% (24% + 33%), and 100% (67% + 33%), respectively, immediately produced correct sums. However, only 48% (5% + 43%), 48% (0% + 48%), 34% (10% + 24%), and 67% (0% + 67%), respectively, immediately produced correct differences.

The upper right-hand cells show the percentages of students who were successful both in addition and in subtraction, and the lower left-hand cells show the percentages of students who were unsuccessful in both. The cells in the upper left-hand corners are important to examine. These cells show that the percentages of students who were successful in subtraction but not in addition were zero or near zero (5%, 0%, 10%, and 0%). The 5% and 10% represent children who were not paying attention when the interviewer presented the addition problems. In contrast, the lower right-hand cells have high percentages of students who were successful in addition but not in subtraction (48%, 48%, 33%, and 33%).

**Figure 1** also shows that when a sum is more difficult to remember (such as 4 + 6), the corresponding difference (10 – 6) also is more difficult.
The first-grade data support our hypothesis: First graders’ knowledge of differences is related to their knowledge of sums.

**Fourth grade**

The fourth-grade data appears in the eight tables in figure 2. Again, many children were successful both in addition and subtraction or were unsuccessful in both. The upper left-hand cells are important to examine. The percentages in these cells are either zero or near zero (0%, 3%, 0%, 0%, 0%, 3%, 11%, and 3%).

By comparing the first two tables in figures 1 and 2, we can conclude that the more solid children’s knowledge of a sum, the easier it is for them to produce a difference. In first grade, almost all the children gave the correct answers within three seconds to 8 + 2 and 6 + 6, but this knowledge was newly constructed and not yet solid. Therefore, only half of the first graders deduced the corresponding differences within three seconds (43% and 48%). By fourth grade, almost all the children deduced the differences with fluency.

With 4 + 6, children’s success rose from 57% (24% + 33%) in first grade to 94% (68% + 26%) in fourth grade, and their success with 10 – 6 rose from 34% (10% + 24%) in first grade to 68% (0% + 68%) in fourth grade. The two tables for 4 + 6 and 10 – 6 show that when children’s knowledge of a sum improves, the corresponding difference becomes easier to deduce.

Among the more difficult sums in fourth grade was 7 + 8. To find the answer, children usually think (7 + 7) + 1, (7 + 3) + 5, or (8 + 2) + 5 (Carpenter et al. 1999; Kamii 2000). Only 53% of the fourth graders gave the correct sum for 7 + 8 within three seconds, and only 32% quickly produced the corresponding difference. This is further evidence that when a sum is more difficult, the corresponding difference also is more difficult to produce.

The fourth-grade data support our hypotheses. If knowledge of differences was independent of knowledge of sums, we would not have found the pattern described above so consistently.

**Discussion**

The reader might say that we have proven that fluency in subtraction depends on fluency in addition but we have not proven that children deduce differences from their knowledge of sums. We can address this objection by comparing the hierarchical thinking involved with addition and subtrac-
tion. As figure 3a illustrates, addition simply involves “ascending” to put two wholes together (5 and 4) to make a higher-order whole (9) of which the original wholes are parts (Inhelder and Piaget 1964). In subtraction (see fig. 3b), by contrast, we start with a whole (9), “descend” to take a part away (5), and “ascend” back to the whole (9) to deduce the other part (4). Therefore, a child who knows the sum of 5 and 4 solidly can deduce quickly the part that is unknown.

The language that we hear in classrooms reflects the difficulty with this thinking at two hierarchical levels. When second graders explain how they added 5 and 4 to compute 15 + 14, for example, their language is straightforward. They might use unusual statements such as the following, however, when they explain how they calculated 9 – 5 to find 29 – 15:

I took away the 5 and the 9.
I subtracted the 5 with the 9.
Five minus 9 is 4.
Five take away 9 equals 4.

Reviewing Piaget’s theory about memory (Piaget and Inhelder 1973) helps clarify how children deduce differences from their knowledge of sums. In Piaget’s theory, the knowledge that 4 + 5 = 9 is a child’s construction from the inside, rather than a fact internalized from the environment, and the recall of 4 + 5 = 9 is a reconstruction of the earlier construction. That is, human memory is not merely the storage and retrieval of facts that have been internalized from the environment. This is why the more solid a sum is in a child’s mind, the more easily the child can reconstruct and use it in deducing a difference. This also explains why certain sums such as 5 + 5 = 10 that are related to the number of fingers and “doubles” such as 2 + 2 = 4 are easier to construct and reconstruct than others such as 5 + 4 = 9 and 4 + 3 = 7.

Educational Implications

The educational implication of this research is that we must de-emphasize fluency in subtraction in the first two grades and heavily emphasize addition. If children know sums very well, they will be able to deduce differences easily from their knowledge of sums. Other implications with respect to counting, word problems, and mathematics games are discussed below.

Counting

Counting to find sums or differences is a procedure that we generally discourage. When a child does not know a sum, counting is the only way that he or she can find it. Counting, however, is a low-level procedure that should give way to thinking. For example, we have observed that children who have constructed doubles, such as 4 + 4 = 8 and 5 + 5 = 10, spontaneously stop counting and begin to use this knowledge. To find 5 + 4, they begin to think (4 + 4) + 1 or (5 + 5) – 1.

We especially want to discourage counting down. Counting down is not only difficult but also
problematic. After saying “Nine-eight-seven-six” to find $9 - 4$, the child has to decide if the answer is 6 (the last counting word) or 5 (the word that comes after “six”). Teachers can eliminate this confusing procedure if they permit students to learn sums first and then deduce differences from their knowledge of sums. If first graders can produce the answer to $10 - 5$ instantly, without any effort or a lesson on subtraction, why should they spend their time learning to count backward?

Counting up from 4 to solve $9 - 4$ is easier than counting down. Counting is still a low-level activity, however, and children’s time is better spent solidly learning sums.

**Word problems**

De-emphasizing mechanical computations in subtraction does not mean that subtraction should be de-emphasized in word problems. In fact, we recommend giving students plenty of word problems involving subtraction, as well as “multiplication” and “division,” because first and second graders and most kindergartners are capable of developing the logic of subtraction, multiplication, and division (Carpenter et al. 1993; Kamii 2000). As figure 4 shows, first graders make correct part-whole relationships to deal with problems such as “We planted 12 pole bean seeds. Seven of them sprouted. How many did not sprout?” Note that this first grader used an additive procedure to answer a “subtraction” problem. Students likewise use addition to solve “multiplication” and “division” problems (Kamii 1990; Kamii and Clark 2000).

**Mathematics games**

Whereas word problems help develop children’s logic of subtraction in real-life situations, mathematics games strengthen their ability to deal with numbers. Therefore, we highlight games such as Find Ten that involve the partitioning of numbers. Find Ten is an addition game that comes close to involving subtraction.

The game uses regular playing cards from 1–9. The object of the game is to find two cards that make 10. All the cards are dealt to three players. Each player keeps his or her cards in a stack, face-down, without looking at them. The players take turns flipping over the top card of their stack and trying to find two cards that make 10. The first player must discard his or her card in the middle of the table, faceup, because making 10 with two cards when only one card is available is impossible. If the second player turns over a 3 and sees a 7...
on the table, he or she can take the 7 and keep the two cards. If the second player turns over any other number, he or she must discard the card faceup in the middle of the table. Play continues until all the cards have been used. The winner is the person who has collected the most cards at the end.

This addition game comes close to involving subtraction because when children turn over a 6, for example, they often say, “I need a 4.” They might ask themselves, “Six plus what makes 10?” For other games involving the partitioning of numbers, see Kamii (2000, pp. 181–89).

Young children avoid subtraction games, but teachers can introduce addition-or-subtraction games such as Sneaky Snake toward the end of first grade. Using the board in figure 5, two players take turns rolling two number cubes to figure out which number they can cover. If a player rolls a 3 and a 5, for example, he or she can cover the 8 (because 3 + 5 = 8) or the 2 (because 5 – 3 = 2).

Children are intrinsically motivated to play games and to play them well. If they learn arithmetic in the process, they learn it for their own use. When teachers instruct students to complete worksheets or pressure them to do well on timed tests, the students’ motivation to learn comes from external sources.

**Next Steps**

We conducted our research with upper-middle-class children, who did their own thinking instead of being taught how to add and subtract by the teacher. We encouraged the children in our study to debate and exchange viewpoints instead of waiting for an indication from the teacher that their answers were correct. The next step in our research is to find out whether children who are taught in traditional ways also deduce differences from their knowledge of sums. We also must conduct research with other socioeconomic groups.

**Comparisons to CGI**

We often are asked how our approach to teaching is similar to and different from Cognitively Guided Instruction (CGI, Carpenter et al. 1999). Both approaches begin with word problems that children solve in a variety of ways they invent, and computational techniques grow out of problem solving. This kind of teaching is different from traditional instruction, which teaches computational techniques first and then gives word problems for children to apply the computational techniques they have learned.

Perhaps one conspicuous difference between CGI and our approach is that CGI’s foundation is the four arithmetical operations and the different problem types related to each operation. For example, one subtraction problem type is separate and another type is compare. An example of the former is “Colleen had 8 guppies. She gave 3 guppies to Roger. How many guppies does Colleen have left?” An example of the latter is “Mark has 3 mice. Joy has 7 mice. Joy has how many more mice than Mark?” (Carpenter et al. 1999, pp. 9–10)

Our approach presents the same kinds of word problems but at different times from CGI, depending on the logic necessary to understand the question. The foundation of our approach is Piaget’s theory of logico-mathematical knowledge, and the view that numerical precision grows out of children’s logic (Piaget 1971). The logic of separating is much easier than that of comparing because the former involves only one whole that students must separate into two parts (Inhelder and Piaget 1964; Kamii 2000). By contrast, comparing involves two wholes (3 and 7 in the preceding problem), the lesser of which students must mentally “transport” onto the greater one and consider a part of the greater whole. Therefore, we present separate problems in first grade but not compare problems. We expect about half of the children in second grade and most of the children in third grade to understand compare problems.

Another difference between CGI and our approach is that CGI does not use mathematics games; we use them frequently. One of our reasons for using games is that children’s cognitive development is inseparably related to their sociomoral development, according to Piaget (1965). For more information on the relationship between sociomoral and intellectual development, see Carpenter et al. (1999) and Kamii (2000).

**References**


Inhelder, Barbel, and Jean Piaget. *The Early Growth of
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