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The Power of Paper-Folding Tasks: Supporting Multiplicative Thinking and Rich Mathematical Discussion

Building understanding of multiplicative relationships is a key goal of mathematics instruction in the upper elementary and middle grades. Multiplicative thinking includes comparing numbers through many processes: multiplication and division (rather than addition and subtraction), ratio, proportions, stretching and shrinking, magnification, scaling, and splitting. Research has shown that multiplicative thinking develops slowly in children, over long periods of time (Clark and Kamii 1996; Vergnaud 1988). Initially, children tend to reason additively about multiplicative situations, and this additive thinking is often resistant to change (Hart 1984). Students need practice with tasks that help develop multiplicative thinking—in particular, tasks that help them recognize and reason about multiplicative relationships.

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Solving and discussing mathematical problems is also an essential part of doing and learning mathematics. Many teachers agree that such discussion is important, but helping students articulate, justify, and debate ideas can be challenging (O'Connor 2001). Teachers need to structure opportunities for students to share their thinking and consider the ideas of others. Although research has shown that all students, including lower-performing and English Language Learner (ELL) students, can and do participate in discussions (see, e.g., Empson 2003; Moschkovich 1999), some students need additional support (see, e.g., Baxter, Woodward, and Olson 2001). Also, not all tasks generate rich mathematical discussions. Students need to reason and communicate about interesting tasks that can be solved in various ways and that lead to important mathematical ideas (Hiebert et al. 1997). In this article, we focus on a particular task—paper folding—and describe its potential to address both these challenges: supporting the development of multiplicative thinking and facilitating discussion of mathematical problems for *all* students.

Paper Folding: Not a Trivial Task

Consider how an elementary school student might reason about the following paper-folding task:

> If I fold a paper in half, in half again, and finally in half again, how many equal parts will I have when the paper is opened? (Do not open the paper between folds.) (See **fig. 1**.)

Although young children can repeatedly fold paper and appreciate the rapid growth in the number of parts, they may not always know how paper fold-

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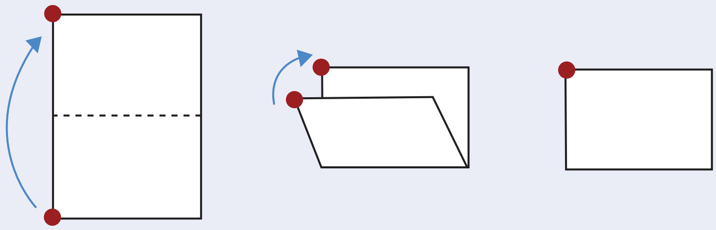
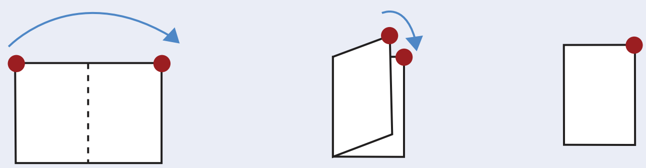
Figure 1

Illustration of paper-folding task

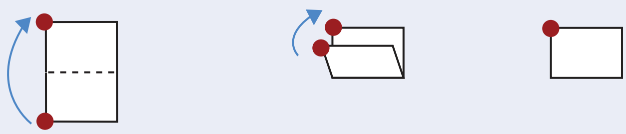
Step 1: Fold an 8 ½-by-11-inch sheet of paper in half.



Step 2: Without unfolding the paper, fold it in half again.



Step 3: Fold the paper in half one more time.



How many equal parts will the sheet of paper have when it is unfolded?

ing "works." Understanding how a sequence of folds results in a certain number of equal parts is not trivial. In fact, a deep understanding requires making sense of the multiplicative relationships embodied in equal folding. For example, each time you fold a paper in half, you double the number of parts, so that folding a paper in half three times creates 2, then 4, and ultimately 8 equal parts ($2 \times 2 \times 2 = 8$).

We presented this task to a cross section of elementary school students—first, third, and fifth graders—with no prior experience in the mathematics of folding. Only four of thirty students correctly predicted that 3 half folds make 8 equal parts. Most ($n = 21$) predicted that 3 half folds would make 6 parts. As one third grader incorrectly reasoned, "If you fold it [in half] once, it's 2 parts, and then again, it's 4 parts, and then one more time, it's 6 parts" because each half fold "adds" 2 more parts to the total.

Children initially tend to reason about paper folding in additive rather than multiplicative ways—that is, as they comprehend the process, folding the paper in half *adds* 2 parts rather than *doubles* the number of existing parts. We decided to provide additional folding tasks for the children to solve and reflect on to help them make sense of how folding works and thus develop their multiplicative thinking. Furthermore, the folding tasks' unique structure—which supports testing, revising, and communicating ideas as well as multiple solution strategies—would lead, we believed, to rich group discussions.

We investigated students' thinking about paper-folding tasks in two contexts. First, we conducted individual problem-solving interviews with the same group of first, third, and fifth graders to examine how children reason about paper folding (Empson and Turner 2006). We used two types of tasks in these interviews. Some tasks asked the students to predict the number of equal parts created by a given series of folds; others asked students to determine a sequence of folds to create a particular number of equal parts (see **fig. 2** for sample folding tasks). Second, we later implemented many of the paper-folding tasks with two classes of fourth and fifth graders who participated in an after-school mathematics program. In the after-school setting, students first solved the folding tasks individually or in small groups, and then the teacher gathered students for a class discussion. In both studies, we worked with students from diverse, low-income urban schools, including many ELL students.

Folding an unmarked paper into a number of equal parts that represent powers of 2—2 parts, 4 parts, 8 parts—can be done quite precisely without a measuring tool. Folds such as thirds usually give only approximate equal areas. In some contexts, such as comparing fractions, a measuring tool may be needed. In contexts where the total number of pieces is the main focus, "rolling up," also known as fan folding, will provide a good approximation.

We highlight three aspects of paper-folding tasks that we found to be especially powerful and then suggest ways that elementary school mathematics teachers might use paper-folding tasks in instruction.

Why Paper-Folding Tasks Are Powerful

Folding tasks support students in testing and revising mathematical ideas

Part of solving and discussing mathematical problems is noticing patterns, making and testing conjectures, and revising ideas as necessary (NCTM 2000). As many teachers know, helping students pose and refine conjectures can be challenging.

Paper-folding tasks provide students a powerful entry point into these practices. All students, even those who may typically struggle with mathematics, can fold and reason about folding. For example, anyone can make a prediction about the outcome of a series of folds. And, as we learned, students almost always have reasons for their predictions, reasons based on noticing patterns and making conjectures, even when their predictions are incorrect. When Alejandra [all students' names in this article are pseudonyms], a first grader, was asked to predict the number of equal parts created by folding a piece of paper in half 4 times, her reply was, "12 parts." She had previously noticed that when a paper folded in 4 parts was folded in half again, it created "4 more" parts, or 8 parts total. She used this pattern to reason (incorrectly) that if the paper is folded in half again, it must make 12 parts, because "every time you fold it [in half], it makes 4 more [parts]." Being able to refer to the folded paper helped Alejandra communicate and articulate her reasoning.

> "Arghh, I thought it was going to be 12!"

Further, the act of operating on the paper itself allows students to test and revise their conjectures about how folding works. Folding tasks are unique in this regard; most mathematical problems do not include a mechanism that permits students to test ideas and receive feedback. Lorena, a third grader, was trying to fold the paper to make 12 equal parts. She first folded the paper repeatedly in half "to see if there is a pattern." Realizing that half folds created 8 parts and 16 parts but never 12 parts, Lorena took a new piece of paper and decided to revise her plan. This time, she started by folding the paper in thirds and then folding it in half; she thought "it might make 6" because 3 plus 3 equals 6. Lorena had a conjecture about folding—"when you fold it in half, it doubles the number of parts"—which she was able to test. When she realized that the half fold doubled the 3 parts to make 6, she opened the paper up to examine the crease lines, refolded it, and then continued with an additional half fold. She explained that the last half fold "might" create 6 more parts for a total of 12, because folding in half seemed to double the total number of parts.

The feedback intrinsic to folding tasks is helpful even to students who begin tasks without a particular theory about folding or without a plan to create a given number of parts. For example, to create 24 parts several students decided to repeatedly fold the paper in half "just to see if it gets to 24." They had the notion that "the more you fold the paper, the more parts you get," but they did not seem to be thinking about the relationship between each fold and the total number of parts. Once the students began folding and counting the parts created by each fold, this built-in feedback helped them generate more refined ideas about folding that they could test and revise. Having a goal in mind—for example, making 24 equal parts—was especially helpful because it focused the students' activity and provided multiple opportunities for them to generate, test, and revise their theories about how paper folding works.

Students' initial theories about folding often reflect additive reasoning—that is, each fold adds a fixed number of parts. But repeated testing of ideas and observations of results helps students make connections and revise their initial understandings (Karmiloff-Smith and Inhelder 1974). In fact, after the third and fifth graders we interviewed worked on several tasks, all but one shifted away from additive reasoning about paper folding. We believe that these advances in student thinking are due in part to the unique structure of folding tasks, which provides immediate feedback about ideas and conjectures.

Folding tasks support students in making powerful mathematical connections

Paper-folding tasks help students make powerful mathematical connections that aid multiplicative thinking. When Mateo, a fifth grader, folded a sheet of paper to make 9 equal parts, he realized that because "3 times 3 is 9," he could fold the paper in thirds and then in thirds again. He recognized that although folding in half doubles the numbers of parts, folding in thirds divides all previously folded parts in 3, so folding in thirds and then in thirds again creates 9 parts ($3 \times 3 = 9$).

Other students make connections between paper folding and multiplication arrays. To fold the paper into 12 parts, Reynaldo, another fifth grader, suggested folding it into 3 columns and then dividing each column into 4 parts, because 3 times 4 equals 12. Reynaldo used an image of a 3×4 array to plan his folding actions and make sense of how folding worked. He realized that when he folded a paper first in 3 parts and then in 4 parts, he was simultaneously dividing each of the 3 columns into 4 parts, thus creating the 3×4 array.

Folding tasks also allow students to think about relationships between factors and products. For instance, to make 24 parts, Andrea reasoned that if she could just "get it [the paper] into an eight," she could fold it in thirds to make 24 parts, because 8 times 3 equals 24. Andrea used 3 half folds to build up the 8 equal parts ($2 \times 2 \times 2$) and then continued with her plan (see **fig. 3**). Essentially, Andrea's folds modeled a prime factorization of 24: $24 = 8 \times 3 = (2 \times 2 \times 2) \times 3$. Jesse, another fifth grader, also used factor-product relationships to solve the related-folds task (see **fig. 2,** problem *f*). Jesse reasoned about multiplicative relationships among the factors of 24 to figure out how Derrick should fold the paper to create the same number (24) of equal parts as Camila did. Although most students used multiplication to solve this problem, Jesse did

Figure 2

Paper-folding tasks

Type of Task	Sample Problem
Predict the result of a series of folds	*a.* If you fold this piece of paper in half 4 times, how many equal parts will you create? *b.* If you fold this paper in half, then in thirds, and then in thirds again, how many equal parts will you create?
Create a given number of equal parts	*c.* Fold this piece of paper to make 6 equal parts. *d.* Fold this piece of paper to make 12 equal parts. *e.* I folded a piece of paper and created 24 equal parts. What steps did I take to lead to 24 equal parts? Would other steps lead to 24 equal parts?
Create a related sequence of folds	*f.* Camila folded a piece of paper into 3 equal parts and then 8 equal parts. Derrick folded his paper into 6 equal parts. If he wants to make exactly as many parts as Camila, into how many parts should he next fold his paper?
Create an "interesting fold"	*g.* You decide into how many parts you want to fold the paper, and you experiment with the paper to see if you can make that many parts. This is your "interesting fold." Record detailed directions for your "interesting fold" so that one of your peers could fold the paper exactly as you did.

Figure 3

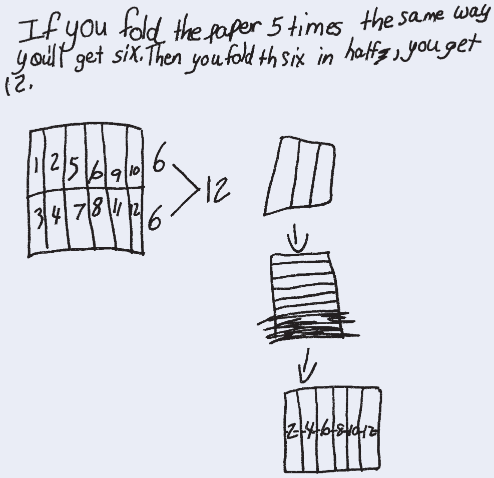
Andrea folding an 8½-by-11-inch sheet of paper to make 24 equal parts

"Folding is like multiplication!"

"Folding is like fractions!"

Figure 4

Pablo's method for making 12 equal parts



"Who found a different way to make 24 equal parts?"

not multiply the first two factors and then find the missing factor in the second pair ($3 \times 8 = 24$, so $6 \times ? = 24$). Instead, he reasoned that because Derrick's first fold (into 6 parts) was twice Camila's first fold (into 3 parts), Derrick had "to do half her second move (into 8 parts)." He reached this solution by thinking about how a fold into 6 equal parts creates twice as many parts as a fold into 3 equal parts ($6 = 3 \times 2$); in other words, this solution is like folding in 3 parts and then folding again in half. Because Derrick created twice as many parts as Camila on his "first move," Jesse reasoned that Derrick's second fold should create half as many parts as Camila's—that is, 4 parts instead of 8: $(3 \times 2) \times 4 = 3 \times (2 \times 4)$.

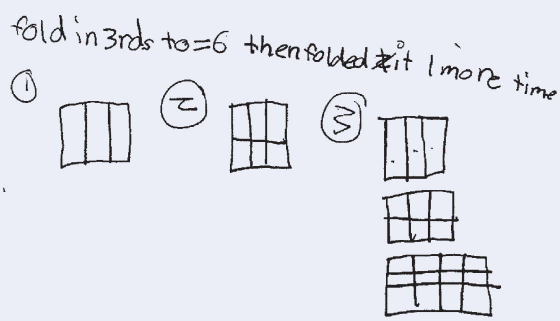
Other students see connections between paper folding and fractions. For example, when Bety, Yasmine, and Paola experimented with repeatedly folding a paper in half, they noticed relationships among the sizes of the parts they created. When the paper had 8 equal parts, they noticed that 2 of those parts (2/8) were equivalent to 1 part on a paper folded in 4 equal parts (2/8 = 1/4). Together they compared the folded papers to identify and justify other equivalent fractions, such as 2/4 = 1/2 and 4/8 = 1/2. Paper-folding tasks also help students reason about the multiplicative relationships among fractions. Later, during a class discussion, Bety, Yasmine, and Paola used the folded papers to explain to their peers that 1/8 is one-half of 1/4 and that 1/4 is one-half of 1/2.

Folding tasks also create opportunities for students to reason about fractions that have different shapes but equal area. When the students discussed ways to make 12 equal parts, they examined a paper folded first in sixths and then in half to create "long, skinny" twelfths; they then compared this paper with a paper folded first in thirds, then in half, and then again in half to create "rectangle-shaped" twelfths (**figs. 4** and **5**). In the discussion that followed, some students realized that although the twelfths "looked different," they had to be equal because each whole was divided into 12 equal parts; therefore, one of the "long skinny" twelfths must be equivalent to one of the "rectangle-shaped" twelfths.

We see tremendous possibilities here. As students work on folding tasks, they make connections to mathematical ideas such as multiplication and fractions. More important, these connections support the development of multiplicative thinking.

Figure 5

Brenda folds in thirds, then in half, and then in half again to make 12 equal parts.



Folding tasks support *all* students in solving and discussing mathematical problems

Although not all tasks generate rich mathematical discussion, we believe there are many reasons why paper folding may be an especially good task for supporting *all* students' participation in communicating, justifying, and debating mathematical ideas.

To begin, although children have folded paper, few have considered how folding produces a particular number of parts. Students' common inexperience about how to solve these problems creates a space where all ideas are welcome. Many share such basic insights as, "I noticed that folding in half adds two parts" (rather than doubles the number of parts), and publicly stating such conjectures helps all students build an understanding of how folding works.

As students articulate their thinking and share their solution strategies with peers (**fig. 6**), having a physical artifact to refer to as they explain can be very helpful. With folding tasks, students can use the paper to demonstrate their plan or prediction. This can be especially helpful for ELL students. During a group discussion, Adán, a Spanish-dominant ELL student in our after-school program, shared his strategy for making 12 equal parts. He explained, "I folded it like this, and like this," modeling his sequence of folds as he talked. The teacher then restated and refined his idea, emphasizing that he first folded the paper "in half" and then "in six parts." Although Adán used ambiguous language in his initial explanation, both the teacher and the other students were able to make sense of his strategy because he demonstrated his folds as he communicated his ideas (**fig. 7**). (See Turner et al. 2006 for additional discussion of how folding tasks supported ELL students' participation in discussion.)

Folding tasks also help generate discussion because many tasks, such as "Fold this paper to make ___ equal parts," support multiple solution strategies. Possible solutions include both basic strategies and more complex strategies that use multiplicative reasoning. For example, to make 12 equal parts, students can fold the paper in half, in half again, and then in thirds, because they know that $2 \times 2 \times 3 = 12$. Another student might know that $2 \times 6 = 12$ and decide to fold the paper in half and then in 6 parts to make 6 parts on each side for a total of 12. Or, even more simply, students can fold the paper over on itself, using a "rolling fold," or fan fold, to create as many parts as necessary. Because multiple strategies exist for making a certain number of parts, including some that do not depend on formal knowledge of multiplication or fractions, all children have the opportunity to contribute ideas to a class discussion.

As individual students present their solutions (**fig. 8**), the teacher can invite the others to compare strategies and look for mathematical relationships to extend their thinking. During a class discussion of the "make 12 equal parts" task, Erika shared her strategy: "I folded [the paper] in half, then in half again, then in thirds." Later, Moana and Natalie shared similar approaches. Moana had folded the paper in half, then in thirds, and then in half again. Natalie had folded the paper in thirds, then in half to make 6 parts, and then in half again. The

Figure 6

Bety explains her solution to the class.

Figure 7

Adán shares his solution with the class.

Figure 8

During a group discussion, Ernesto shares his strategy for a folding task.

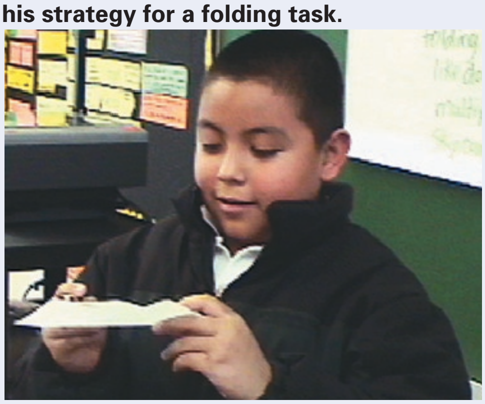
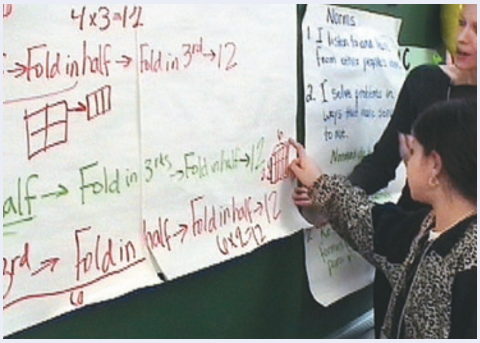


Figure 9

Bety compares strategies for making 12 equal parts.



teacher asked the class to compare the strategies. Bety noted that they all involved two half folds and a third fold, just in different order (**fig. 9**). Other students agreed that "[the folds] are just reversed" and that "the order doesn't matter, it's always 2 and 3 and 2, and that makes 12 [parts]." The teacher helped the students record the folds numerically—$2 \times 3 \times 2 = 12$; $2 \times 2 \times 3 = 12$; $3 \times 2 \times 2 = 12$; $6 \times 2 = 12$—thus creating another opportunity for students to discuss the mathematical relationship among the different strategies.

Suggestions for Paper Folding in the Elementary Classroom

How do I start?

Try some of the problems listed in **figure 2**. Explore the tasks yourself, and then try them with your students. Use the activities as a context for making and testing predictions and making conjectures about how paper folding works. Give students plenty of time to experiment. The "fold the paper to make [a certain number of] parts" tasks are especially powerful because they provide students with multiple opportunities to test and revise their ideas.

What kind of paper should I use?

We used patty paper for all the folding tasks. Patty paper is square-shaped, thin, waxy paper that is easily available through many mathematics educational supply catalogues. It is easy to fold, and the lines made by the folds are easily distinguishable.

What kinds of questions should I ask?

Ask questions to discern your students' thinking. Such questions could include the following:

- "Why do you think the folded paper or the creases resulting from the folds turned out that way instead of the way you predicted?"
- "Before you start folding, tell me about your plan for making 6 equal parts. Why do you think that will work?"
- "Now you have 4 parts. If you fold the paper in half again, what do you think will happen?"
- "What patterns have you noticed with folding?"
- "You said that folding in half and then in half again is like multiplication, because it's like 2 times 2. Tell me more. How is this like multiplication?"

Ask questions to discern your students' strategies. Such questions could include the following:

- "What did you do first? What did you do next?"
- "Can you draw each step for me?"
- "How is your folding strategy like or not like [another student's] strategy?"

Other notes about instruction

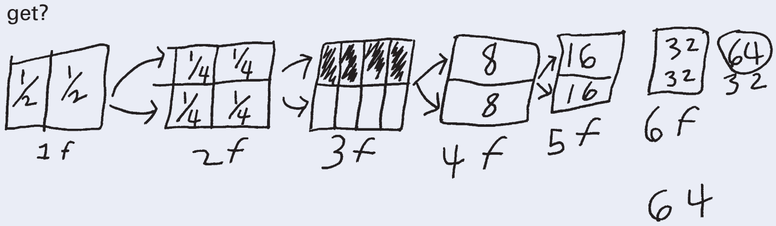
To encourage children to relate folding to multiplication, you might use language such as this: "If you make 3 equal *parts* and then 4 equal *parts*, how many equal *parts* will you have when you open up your paper?" To encourage students to relate folding to fractions, use language such as this: "If you fold in *thirds* and then in *fourths,* what fraction will your paper show when you open it?" Making connections between fractions and multiplication may be an issue that naturally comes up, but in either case you might ask these questions:

- "How is folding like multiplication? Why?"
- "How is folding like fractions? Why?"
- "How is folding like division? Why?"

Figure 10

Ramón represents the outcome of repeatedly folding in half.

If you fold this piece of paper in half 6 times, how many equal parts will you get?



Encourage students to write out their procedure. Students may use a combination of words, pictures, fractions, and multiplication facts to show how they folded the paper (**fig. 10**).

Use students' representations to make connections between different solution strategies. Discussing different representations is a powerful mathematical activity.

Conclusion

Paper folding provides powerful mathematical tasks that help students articulate reasoning, make and revise conjectures, and develop multiplicative reasoning. In addition, these tasks are engaging for both students and teachers. The act of folding paper itself creates a shared space that enhances teacher-student and student-student communication and facilitates group discussion. Folding tasks like the ones we describe here are rich with opportunities for teachers to face the challenge of engaging *all* students in solving and discussing mathematical problems.

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